

SERIES OF RECIPROCAL POWERS OF k -ALMOST PRIMES

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ABSTRACT. Sums over inverse s -th powers of semiprimes and k -almost primes are transformed into sums over products of powers of ordinary prime zeta functions. Multinomial coefficients known from the cycle decomposition of permutation groups play the role of expansion coefficients. Founded on a known convergence acceleration for the ordinary prime zeta functions, the sums and first derivatives are tabulated with high precision for indices $k = 2, \dots, 6$ and integer powers $s = 2, \dots, 8$.

1. OVERVIEW

Series over rational polynomials evaluated at integer arguments contain sub-series summing over integers classified by the count of their prime factors. The core example is the Riemann zeta function ζ which accumulates the prime zeta function P_1 plus what we shall define the almost-prime zeta functions P_k (Section 2). The central observation of this manuscript is that the almost-prime zeta functions are combinatorial sums over the prime zeta function (Section 3). Since earlier work by Cohen, Sebah and Gourdon has pointed at efficient numerical algorithms to compute P_1 , series over reciprocal almost-primes—which may suffer from slow convergence in their defining format—may be computed efficiently by reference to the P_1 .

In consequence, any converging series over the positive integers which has a Taylor expansion in reciprocal powers of these integers splits into k -almost prime components. Section 4 illustrates this for the most basic formats.

Number theory as such will not be advanced. The meromorphic landscape of the prime zeta functions as a function of their main variable appears to be more complicated than what is known for their host, the Riemann zeta function; so only some remarks on the calculation of first derivatives are dropped.

2. PRIME ZETA FUNCTION

Definition 1. *The prime zeta function $P(s)$ is the sum over the reciprocal s -th powers of the prime numbers p [16, 15, 27]*

$$(1) \quad P(s) \equiv \sum_p \frac{1}{p^s}; \quad \Re s > 1.$$

Remark 1. *The primes are represented by sequence A000040 in the Online-Encyclopedia of Integer Sequences [32], and we will adopt the nomenclature that a letter A followed by a 6-digit number points at a sequence in this database. Accurate values*

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$P(s)$ for $s \leq 9$ are provided by the sequences A085548, A085541, and A085964–A085969.

Table 1 complements these by using [4, 30, 13]

$$(2) \quad P(s) = \sum_{p \leq M} \frac{1}{p^s} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log P(M, sn)$$

for a suitably large prime M , where μ is the Möbius function [1, (24.3.1)], where ζ is the Riemann zeta function [11, 22][18, (9.5)], and

$$(3) \quad P(M, s) \equiv \zeta(s) \prod_{p \leq M} (1 - p^{-s})$$

an associated definition of a partial product.

Moments are listed in Table 2. The first line,

$$(4) \quad \sum_{s=1}^{\infty} \frac{1}{s} P(s) = - \sum_p \left[\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right] = \lim_{s \rightarrow 1} [\zeta(s) - \sum_p \frac{1}{p^s}],$$

is A143524 [32, 13]. The total of all values in Table 2 is

$$(5) \quad \begin{aligned} \sum_{u=1}^{\infty} \sum_{s=2}^{\infty} \frac{1}{s^u} P(s) &= \sum_{s=2}^{\infty} \sum_p \frac{1}{(s-1)p^s} = - \sum_p \frac{\log(1 - 1/p)}{p} \\ &= 0.58005849381391172358283349737677118691587319037 \dots \end{aligned}$$

The first derivatives $dP(s)/ds$ of (1) are evaluated as the first derivatives of (2) [1, (3.3.6)],

$$(6) \quad P'(s) = - \sum_p \frac{\log p}{p^s} = - \sum_{p \leq M} \frac{\log p}{p^s} + \sum_{n=1}^{\infty} \mu(n) \frac{P'(M, sn)}{P(M, sn)};$$

$$(7) \quad \frac{P'(M, s)}{P(M, s)} = \frac{\zeta'(s)}{\zeta(s)} + \sum_{p \leq M} \frac{\log p}{p^s(1 - p^{-s})}.$$

Here, primes denote derivatives with respect to the main argument, which is the second argument for the case of $P(.,.)$. Table 3 shows some of them for selected small integer s .

Remark 2. The relation $-\zeta'(s)/\zeta(s) = \sum_p \log(p)/(p^s - 1)$ is a gateway to acceleration of series involving logarithmic numerators. Examples are growth rate coefficients of unitary square-free divisors [33],

$$(8) \quad \begin{aligned} \sum_p \frac{(2p+1) \log p}{(p+1)(p^2+p-1)} &= \sum_p \log p \left[\frac{2}{p^2-1} - \frac{3}{p^3-1} + \frac{4}{p^4-1} - \frac{10}{p^5-1} + \frac{18}{p^6-1} \right. \\ &\quad \left. - \frac{28}{p^7-1} + \frac{40}{p^8-1} - \frac{72}{p^9-1} + \dots \right] = 0.748372333429674 \dots \end{aligned}$$

or of unitary cube-free divisors,

$$(9) \quad \begin{aligned} \sum_p \frac{(4p^2-2-p) \log p}{(p^2-1)(p^2+p-1)} &= \sum_p \log p \left[\frac{4}{p^2-1} - \frac{5}{p^3-1} + \frac{7}{p^4-1} - \frac{17}{p^5-1} + \frac{31}{p^6-1} \right. \\ &\quad \left. - \frac{48}{p^7-1} + \frac{69}{p^8-1} - \frac{124}{p^9-1} + \dots \right] = 1.647948081159756 \dots \end{aligned}$$

TABLE 1. The Prime Zeta Function of some integer arguments. In a style adopted from [1], optional trailing parentheses contain an additional power of 10. Example: The number 3.45×10^{-3} may appear as 0.00345 or as $3.45(-3)$ or $.345(-2)$. Trailing dots indicate that more digits are chopped off, not rounded, at the rightmost places.

s	$P(s)$
10	.9936035744369802178558507001477394163018725452852033205535666(-3) ...
11	.4939472691046549756916217683343987121559397009604952181866074(-3) ...
12	.2460264700345456795266485921650809279799322679473231921741459(-3) ...
13	.1226983675278692799054887924314033239147428525577690135256528(-3) ...
14	.6124439672546447837750803429987454197282126872378013541885303(-4) ...
15	.3058730282327005256755462931371262800130114525389809330765981(-4) ...
16	.1528202621933934418080192641189055977466126987760393110788060(-4) ...
17	.7637139370645897250904556043939762017569839042162662520251345(-5) ...
18	.3817278703174996631227515316311091361624942636382614195748077(-5) ...
19	.1908209076926282572186179987969776618145616195068986381165765(-5) ...
20	.9539611241036233263528834939770057955700555885822134364992986(-6) ...
21	.4769327593684272505083726618818876106041908102543778311286107(-6) ...
22	.2384504458767019281263116852015955086787325069914706736138961(-6) ...
23	.1192199117531882856160246453383398577108304116591413496750467(-6) ...
24	.5960818549833453297113066655008620131146582480117715598724992(-7) ...
25	.2980350262643865876662659401778145949592778827831139923166297(-7) ...
26	.1490155460631457054345907739442373384026574094003717094826319(-7) ...
27	.7450711734323300780164546124093693349559346148927152517177541(-8) ...
28	.3725334010910506351833912287693071753007133176180958755544001(-8) ...
29	.1862659720043574907522145113353601172883347161316571090677067(-8) ...
30	.9313274315523019206770664589654477590951135917359845054142758(-9) ...
31	.4656629062865372188024756168924550748371110904683071460848803(-9) ...
32	.2328311833134403149136721429290134383956012839353792695549588(-9) ...
33	.1164155017134526496600716286019717301900642951759025845555278(-9) ...
34	.5820772087563887361296110329279891461135544371461870778050157(-10) ...
35	.2910385044412396334030528313212481809718543835635609582681388(-10) ...
36	.1455192189083022590216132905087468529073564045445529854681582(-10) ...
37	.7275959835004541439158484817671131286009806802799652884873652(-11) ...
38	.3637979547365416297743239172591421915260411920812352913301657(-11) ...
39	.1818989650303757224685763903905856620025233007531508321634193(-11) ...

3. ZETA FUNCTIONS OF ALMOST-PRIMES

3.1. Nomenclature. We generalize the notation, and define the k -almost prime zeta functions by summation over inverse powers of k -almost primes q_j ,

Definition 2.

$$(10) \quad P_k(s) \equiv \sum_{j=1}^{\infty} \frac{1}{q_j^s}.$$

TABLE 2. Moments of the Prime Zeta function. The difference between the first two lines has been quoted by Chan [10].

u	$\sum_{s=2}^{\infty} P(s)/s^u = \sum_{s=2}^{\infty} \sum_p 1/(s^u p^s)$
1	.31571845205389007685108525147370657199059268767872439261370 ...
2	.139470639611308681803077672937394594285963216485548292900601 ...
3	.646081378884568610840336242581546273479300400893390674692107(-1) ...
4	.307990601127799353645768912808351277991590439776491610562576(-1) ...
5	.149414104078444272503589211881003770331602105485328925740449(-1) ...
6	.732763762441199457647551708167392749211019881968761942480213(-2) ...

TABLE 3. The first derivative of the Prime Zeta Function at some integer arguments s . The value in the first line differs from Cohen's value [13] after 42 digits.

s	$P'(s)$
2	-4.930911093687644621978262050564912580555881263464682907133271(-1) ...
3	-1.507575555439504221798365163653429195755011615306893318187976(-1) ...
4	-6.060763335077006339223098370971337840638287746125984399112768(-2) ...
5	-2.683860127679835742218751329245015994333014955355822812481980(-2) ...
6	-1.245908072279999152702779277468997004091135047157587587410933(-2) ...
7	-5.940689039148196142550592829016609019368189505929351075166813(-3) ...
8	-2.879524708729247391346028423857334064998983761675865841067618(-3) ...
9	-1.410491921424531291554196456308199977901657131693496192836500(-3) ...
10	-6.956784473446204802000701977708415913844863703329838954712256(-4) ...
11	-3.446864256305149016520798301347221055148509398720732052598028(-4) ...
12	-1.712993524462175657532493112138275372004981118241302276420951(-4) ...
13	-8.530310916711056635208876017215691972617326615054214472499073(-5) ...
14	-4.253630557412291035554757415368617516720893534438843631304558(-5) ...
15	-2.122979056274934599669348621302375720453112762226994727150844(-5) ...
16	-1.060211861676127903320578231686279299852887328732516230264968(-5) ...
17	-5.296802557643848074496697331902062291354582070044729659167083(-6) ...
18	-2.646982787802997352263261854182101806956865359404392741570106(-6) ...
19	-1.323018648512292735443206851957658773372595611301942028763990(-6) ...
20	-6.613517594172600210891457029052100560435779681754585968634604(-7) ...
21	-3.306233614825208657730023089591286331399889373034673500892396(-7) ...
22	-1.652941753425972669328543237067224505237754606957895309294978(-7) ...
23	-8.264125267365738127779160862943622945349018242909935277495509(-8) ...
24	-4.131868136465068742054546598016395808846430264964223880702300(-8) ...
25	-2.065869236367122379085627896761801003317031762395199472960982(-8) ...
26	-1.032913007669833840610060139473968867069681131430230354135307(-8) ...
27	-5.164493003519525183949097124602884025560927707601929381567114(-9) ...
28	-2.582222490193098373680412778703362364263231350157972835761708(-9) ...
29	-1.291103241249637065884459649285079993129747526229207669548247(-9) ...

In slight violation of the almost-terminology, the Prime Zeta Function is incorporated as just one special case,

$$(11) \quad P_1(s) \equiv P(s).$$

Remark 3. The sequence q_j is given by the primes A000040 if $k = 1$, by the semiprimes A001358 if $k = 2$, by the 3-almost primes A014612 if $k = 3$, by the 4-almost primes A014613 if $k = 4$, by the 5-almost primes A014614 if $k = 5$, by the 6-almost primes A046306 if $k = 6$, by the 7-almost primes A046308 if $k = 7$ etc. $P_2(2)$ is A117543, and $P_3(2)$ is A131653.

Remark 4. One step further defines prime multi-zeta functions of the form

$$(12) \quad P(s_1, s_2, \dots, s_k) \equiv \sum_{p_1, p_2, \dots, p_k=2,3,5,\dots} \frac{1}{p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}}.$$

If the exponents are restricted to a common number s , they retrieve the information of the $P_k(s)$ in slightly entangled form, for example $P(s, s) = 2P_2(s) - P(2s)$ [8].

Each integer $n > 0$ is either a member of the set $\{1\}$, or of the set of primes, or of the set of semi-primes, etc. These disjoint sets are labeled by the sum of the exponents of the prime number factorizations of their members,

$$(13) \quad \Omega(n) = k; \quad \Omega(1) \equiv 0.$$

The Riemann zeta function may be partitioned into sums over the almost-prime zeta functions,

$$(14) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} = 1 + \sum_{k=1}^{\infty} P_k(s).$$

Remark 5. $\zeta(s)$ may be taken from [1, (Table 23.3)] for $s \leq 42$, or from A013661, A002117, A013662 – A013678 while $s \leq 20$, or from the link to “Recent additions of tables” in Plouffe’s database [28] while $s \leq 99$. $\Omega(n)$ is tabulated in A001222.

3.2. Numerical Scheme. Table 4 of $P_k(s)$ is deduced from

Theorem 1. [21]

$$(15) \quad P_2(s) = \frac{P(2s) + P^2(s)}{2!};$$

$$(16) \quad P_3(s) = \frac{2P(3s) + 3P(2s)P(s) + P^3(s)}{3!};$$

$$(17) \quad P_4(s) = \frac{6P(4s) + 8P(3s)P(s) + 3P^2(2s) + 6P(2s)P^2(s) + P^4(s)}{4!};$$

$$(18) \quad P_k(s) = \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\dots+kk_k=k \\ k_k \geq 0}} (k; k_1 k_2 \dots k_k)^* P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks),$$

utilizing values of the prime zeta function as discussed above and summing over the partitions $\pi(k)$ of k with weight coefficients

$$(19) \quad (k; k_1 k_2 \dots k_k)^* = k! / \prod_{m=1}^k (m^{k_m} k_m!)$$

of Table 24.2 in [1] (multinomials M_2) and A036039 or A102189.

Proof. (18) is the main result of the paper. For small k , explicit verification can be done along Price's [29] construction, where the k -almost primes fill triangular, ($k = 2$), tetrahedral ($k = 3$) etc. sections of a k -dimensional Euclidean lattice labeled by the prime numbers along its Cartesian axes [5]. The case $P_1(s) = P(s)$ just repeats the definition (10). The case (15) accumulates in $P(2s)$ the sum over the squares, and in $P^2(s)$ —with the binomial expansion— again the sum over the squares and twice the sum over products of distinct primes. After division through $2!$, each semiprime is effectively represented once.

The generic proof follows through induction: the terms of the right hand side of (18) contain the factor $(k; k_1 k_2, \dots k_k)^*$, which is the number of distinct permutations with k_m cycles of length m for $m = 1, 2 \dots k$ [6, (p. 123)][1, (24.1.2)]. The right hand side is the cycle index

$$(20) \quad Z(S_k) \equiv \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\dots+k k_k=k \\ k_k \geq 0}} (k; k_1 k_2 \dots k_k)^* P^{k_1}(s) P^{k_2}(2s) \dots P^{k_k}(ks)$$

of the symmetric group S_k [20, (2.2.5)] with $P(ms)$ substituted for the indeterminates of cycle length m . Skipping any interpretation within a Redfield-Pólya symmetry, its recurrence

$$(21) \quad Z(S_k) = \frac{1}{k} \sum_{j=1}^k P(js) Z(S_{k-j})$$

is already established [20, (2.2.9)]. This matches precisely the recurrence on the left hand side which generates P_k by a combination of products of lower-indexed almost-primes,

$$(22) \quad P_k(s) = \frac{1}{k} \sum_{j=1}^k P(js) P_{k-j}(s).$$

This recurrence is valid because each k -almost prime which appears on the left hand side of this equation can be generated in k ways by a product of the form $P(js)P_{k-j}(s)$: in $\omega(k)$ ways by splitting off a prime number and multiplication with a number of the sum in $P_{k-1}(s)$, for each divisor of the k -almost prime which is a square of some prime in addition by multiplication of the square with a term in the sum in $P_{k-2}(s)$, and so on for divisors that are cubes of some prime etc. \square

The simple pole of $\zeta(s)$ at $s - 1$ with Stieltjes constants γ_j [12, 3, 23],

$$(23) \quad \zeta(s) = \frac{1}{s-1} + \gamma + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \gamma_j (s-1)^j,$$

is associated with a logarithmic singularity of $P(s)$ at $s = 1$ and singularities of $P(s)$ on the real line between $s = 0$ and $s = 1$ where s is the inverse of a square-free integer [16]. If $k > 1$, the k -almost zeta functions inherit these and add more by the mechanism evident from the multipliers in Theorem 1. $P_2(s)$ in (15), for example, inherits singularities at $1/2, 1/3, 1/5, 1/6, 1/7, 1/10$ etc. from the term $P^2(s)$, and singularities at $1/2, 1/4, 1/6, 1/10, 1/12$ etc. from the term $P(2s)$, illustrated in Fig. 1.

TABLE 4. Almost-prime zeta functions $P_k(s)$ at small integer arguments s .

k	s	$P_k(s)$
2	2	1.407604343490233882227509254138772537749192760048802639241489(-1) ...
2	3	2.380603347277195967869595585283620062893217848034845684562765(-2) ...
2	4	4.994674468637339635276874049579289322502057848230867728509096(-3) ...
2	5	1.136012424856354766515556190735772665693748056026108556151424(-3) ...
2	6	2.687071675614096324217387396140875535798787447125719642936101(-4) ...
2	7	6.493314175691145578854061507836714519989975167152833237216508(-5) ...
2	8	1.588851988525958888572372095351879234527858971327233300748108(-5) ...
3	2	3.851619298269464091283792262806039543890016747838157193719155(-2) ...
3	3	3.049362082334312946748098847079302999848694548619577993637287(-3) ...
3	4	3.144274968329417421821246641907192073071706953574340102524412(-4) ...
3	5	3.557725337068269111888017622799305930206716602282958084700573(-5) ...
3	6	4.201275533960671214387834295923202794959720951879928447823823(-6) ...
3	7	5.073887994515979227127878654920650441797124899213497891489656(-7) ...
3	8	6.206813624161469945551964458392656691354524774013736254471908(-8) ...
4	2	1.000943620148325082041084351808525466652473851036634849174401(-2) ...
4	3	3.839045346157269074628008425162843300890790106333110559279434(-4) ...
4	4	1.967963362818191467940855961573410955099879950428233958199339(-5) ...
4	5	1.112105498394147042065416843932409614810339288829649464410277(-6) ...
4	6	6.564866966272364593992630942917565336419279606893244510336952(-8) ...
4	7	3.964020093813893558567748245375642870705531500802782383854435(-9) ...
4	8	2.424542067198129719213221460827573885198415530509018971808441(-10) ...
5	2	2.545076168069302058221776985605516223099431333404435645812102(-3) ...
5	3	4.808940110832567973019045453666287670709263774628437825310148(-5) ...
5	4	1.230321747728495443208363890849979176133316153001252000782537(-6) ...
5	5	3.475459860092756789327837058184938607371038782365590655202548(-8) ...
5	6	1.025765593034930528602801778254441805529589443170682127946500(-9) ...
5	7	3.096892760390829520074774635913281487435638340106655991757766(-11) ...
5	8	9.470868287557099531707292414885674011014480791459591532172493(-13) ...
6	2	6.410338528642807128627067320846767912178898525413482485882042(-4) ...
6	3	6.014928780179108948186295382866155223857573977633433216567001(-6) ...
6	4	7.689936414615761724089452218863183766130186637100554867671046(-8) ...
6	5	1.086086563383684175516301346294503063069865950783005109331817(-9) ...
6	6	1.602759442759730816486790711011122437362717548888087849039669(-11) ...
6	7	2.419447563344257658856721989501609732074582033968956770368419(-13) ...
6	8	3.699557937592796079194223671580493130944594965925089150589532(-15) ...

Remark 6. Sums over odd indices are [31, 2]:

$$(24) \quad \sum_{k=1}^{\infty} P_{2k-1}(2) = \frac{\pi^2}{20}; \quad \sum_{k=1}^{\infty} P_{2k-1}(2s) = \frac{\zeta^2(2s) - \zeta(4s)}{2\zeta(2s)}.$$

Sums over even indices follow from there as the complement in (14).

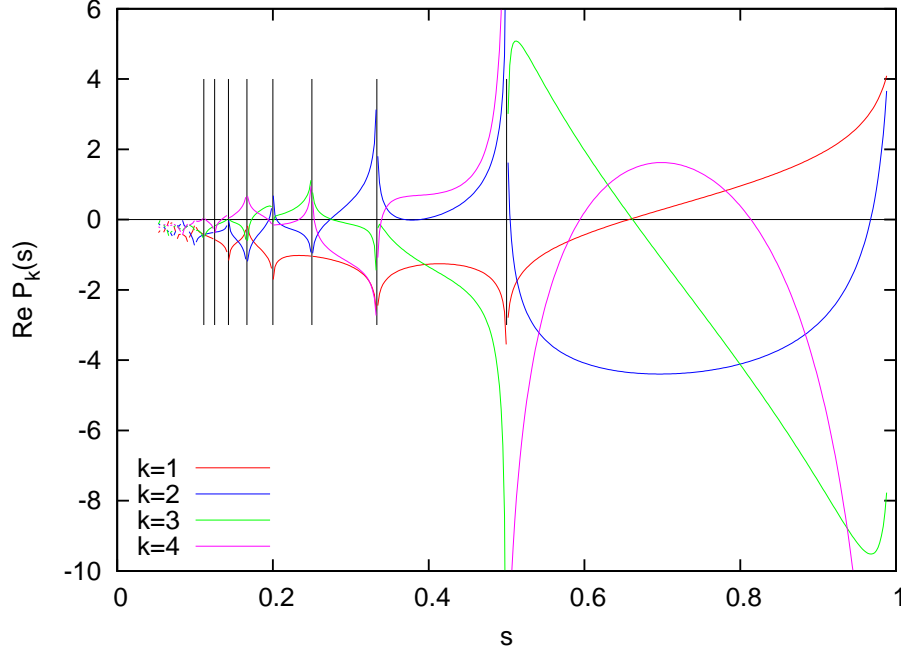


FIGURE 1. Structure of poles and logarithmic singularities: $\Re P_k(s)$ on the real line between 0 and 1.

Corollary 1. *The product rule [1, (3.3.3)] applied to (18) yields the first derivative*

$$(25) \quad P'_k(s) = \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\dots+k k_k=k \\ k_k \geq 0}} (k; k_1 k_2 \dots k_k)^* P^{k_1}(s) P^{k_2}(2s) \dots P^{k_k}(ks) \\ \times \left[\frac{k_1 P'(s)}{P(s)} + \frac{2k_2 P'(2s)}{P(2s)} + \dots + \frac{k k_k P'(ks)}{P(ks)} \right].$$

Numerical evaluation yields Table 5. The underivative $\int_x^\infty P_k(s) ds = \sum_i 1/[p_i^x \log p_i]$ has been evaluated numerically for $k = 1$ by Cohen [13, 25].

3.3. Möbius (Square-free) Variant. Reduction of the summation to k -almost primes with k distinct prime factors defines a signed variant of the prime zeta functions:

Definition 3.

$$(26) \quad P_k^{(\mu)}(s) \equiv \sum_{\substack{n=1 \\ \Omega(n)=k}}^{\infty} \frac{\mu(n)}{n^s} = (-1)^k \sum_{\substack{n=1 \\ \Omega(n)=\omega(n)=k}}^{\infty} \frac{1}{n^s}; \quad P_1^{(\mu)}(s) = -P(s),$$

where $\omega(\cdot)$ is the number of distinct prime factors of its argument.

Remark 7. *The criterion $\Omega(n) = \omega(n) = k$ selects the prime number products (square-free k -almost primes) of A000040 ($k = 1$), A006881 ($k = 2$), A007304 ($k = 3$), A046386 ($k = 4$), A046387 ($k = 5$), A067885 ($k = 6$), A123321 ($k = 7$), A123322 ($k = 8$), A115343 ($k = 9$), etc.*

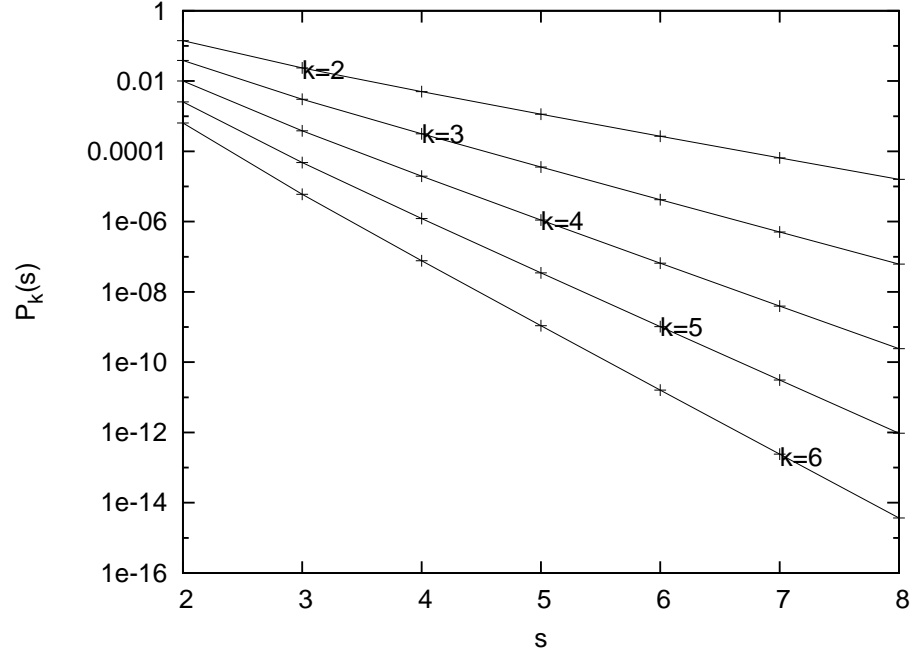


FIGURE 2. A synopsis of table 4 on a semi-logarithmic scale, indicating that the $P_k(s)$ fall off approximately exponentially $P_k(s) \sim 2^{-ks}$ as $s \rightarrow \infty$ along the real s -axis.

The sum

$$(27) \quad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{k=1}^{\infty} P_k^{(\mu)}(s)$$

converges for $s > \frac{1}{2}$ if the Riemann hypothesis holds [14]. Application of the multinomial expansion [1, (24.1.2)] to the powers $P^k(s)$ leads to the recurrences

Theorem 2.

$$(28) \quad P_2^{(\mu)}(s) = \frac{P^2(s) - P(2s)}{2!};$$

$$(29) \quad -P_3^{(\mu)}(s) = \frac{P^3(s) - 3P(s)P(2s) + 2P(3s)}{3!};$$

$$(30) \quad P_4^{(\mu)}(s) = \frac{P^4(s) - 6P^2(s)P(2s) + 3P^2(2s) + 8P(s)P(3s) - 6P(4s)}{4!};$$

$$(31) \quad P_k^{(\mu)}(s) = \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\dots+k_k=k \\ k_k \geq 0}} (-1)^m(k; k_1 k_2 \dots k_k)^* \times P^{k_1}(s) P^{k_2}(2s) \dots P^{k_k}(ks),$$

where $m \equiv k_1 + k_2 + k_3 + \dots + k_k$.

TABLE 5. First derivatives $P'_k(s)$. $P'_1(s)$ is Table 3.

k	s	$P'_k(s)$
2	2	-2.836068154079806522242582225482783360793505782378140134111118(-1) ...
2	3	-3.880586902399322336692460182658731189674126497852135409952156(-2) ...
2	4	-7.545896694085315206907970667196350193021639416359677908501448(-3) ...
2	5	-1.655293105240617640761013220868097514629396965866818280047120(-3) ...
2	6	-3.839769424635045625755371800119456139211625839983225237893684(-4) ...
2	7	-9.174798062469143952346462602570231884264128810420282594760147(-5) ...
2	8	-2.229703572181493732352240313021672944986858675507693036222976(-5) ...
3	2	-1.092764452688696718233957044460372231874277602428901489109438(-1) ...
3	3	-7.176813165338438143871896571868137568859537620262992178861887(-3) ...
3	4	-6.957183997016348677998754615917611673908374611362524278817441(-4) ...
3	5	-7.659277012695409306374743110808079101898869002639397513312088(-5) ...
3	6	-8.918921902960271370285096859450098504725826286458231248526033(-6) ...
3	7	-1.068734610688718635883673494132633669094248021103907624353475(-6) ...
3	8	-1.301295684059645175221229018448850667367013695078510484904330(-7) ...
4	2	-3.603726094351798848506626656181111241130836664796955962932855(-2) ...
4	3	-1.174116309572987946977816010618872204640816441822335628644463(-3) ...
4	4	-5.722998858912958006017304600902369401289250874897789676420392(-5) ...
4	5	-3.165566369796062449665250347230914962435474993776593819875806(-6) ...
4	6	-1.848763022618552000611470797190861080535660685413458944944226(-7) ...
4	7	-1.109730619419583432307793855949753251494307851245851961690094(-8) ...
4	8	-6.763771157059229099811578675678873194164157816965106106489574(-10) ...
5	2	-1.102162098505070183104131920053734921658299916521679210977474(-2) ...
5	3	-1.806134929387963117989216971707723297041186986588483316621559(-4) ...
5	4	-4.431364427680593899920896902337095946811075675701701222137028(-6) ...
5	5	-1.23020326375879194269629288431245922727740434277696211594902(-7) ...
5	6	-3.599723532708191784184538837522982283120808246510836532161742(-9) ...
5	7	-1.081638288290161011509961768322567652158899297605137495433586(-10) ...
5	8	-3.298569076163768263274720121521048493116765743835081292530352(-12) ...
6	2	-3.232720312150523118304098243969541303542841741356062914953186(-3) ...
6	3	-2.676915386444316803744871335276940423982308402686133969144141(-5) ...
6	4	-3.302884682590606781698851083367477685502328729832308044821381(-7) ...
6	5	-4.597234953883457819527050674957592810761129310502451413327254(-9) ...
6	6	-6.735520381085029586780892400600876281099743200915951990514023(-11) ...
6	7	-1.012733294209721415331392073627541785560477001434410873508589(-12) ...
6	8	-1.544937368294918275696159095802943943066554391662366563388540(-14) ...

Redistributing the sign with

$$(32) \quad (-1)^m P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks) = P_1^{(\mu)k_1}(s) P_1^{(\mu)k_2}(2s) \cdots P_1^{(\mu)k_k}(ks),$$

shows that a recurrence equivalent to (21) is applicable.

Remark 8. Sums over odd indices are [31]:

$$(33) \quad \sum_{k=1}^{\infty} P_{2k-1}^{(\mu)}(2) = -\frac{9}{2\pi^2}; \quad \sum_{k=1}^{\infty} P_{2k-1}^{(\mu)}(2s) = -\frac{\zeta^2(2s) - \zeta(4s)}{2\zeta(2s)\zeta(4s)}.$$

TABLE 6. Almost-prime zeta functions $P_k^{(\mu)}(s)$ at small integer arguments s computed from (31).

k	s	$P_k^{(\mu)}(s)$
2	2	6.37672945847765432801316294807193836128782162900370736592109(-2) ...
2	3	6.73594662213544672456228258677680141934623660580421211246428(-3) ...
2	4	9.33269102119509074753434906896208799524913336159126727476407(-4) ...
2	5	1.42408850419374548659705490588033249391875510740905235597858(-4) ...
2	6	2.26806975268639528950901474490066255999464767652487721194642(-5) ...
2	7	3.68874503144697741103258077849260322707848294774819695331204(-6) ...
2	8	6.06493665920244704921794541628232570617319835668401899600473(-7) ...
3	2	-3.6962441634528353783955346323946681155915397130304272497472(-3) ...
3	3	-6.6148651246349939521729829639111115641021894727404106069829(-5) ...
3	4	-1.7271458093722304630212588271041732572671841808663978769400(-6) ...
3	5	-5.0940194598826356852005108113237778072664477868404106866961(-8) ...
3	6	-1.5823229154549293389076239250147682789572853572784414723597(-9) ...
3	7	-5.0453603114670647581939240532674248065812561949214992614300(-11) ...
3	8	-1.6329431236938215954403416483738501457291953254025603265554(-12) ...
4	2	1.05117508492309807485233009466098526324680558243958672947068(-4) ...
4	3	2.14173193213549705893739943930728906255490278218470044772058(-7) ...
4	4	7.29603168874401925790854604647164607340676506339572130564981(-10) ...
4	5	2.96196721173369084753821609237130107261806114215416577935264(-12) ...
4	6	1.29842711892568424473824206373082938247800573758362639862211(-14) ...
4	7	5.91005736941452577777835874048660952776699696196276661514317(-17) ...
4	8	2.74348813375914336305677937087519697186480294957450635783864(-19) ...
5	2	-1.6620822035796812822471192427038246964759250664122544710020(-6) ...
5	3	-2.6408478825460477590567284836043051634661147998573638254764(-10) ...
5	4	-7.6296745513152797837319954330917616771026320680788925154580(-14) ...
5	5	-2.6674855904890302723828983961667330621935895399254169680514(-17) ...
5	6	-1.0124771878309455119843894285163727048400779511505829601925(-20) ...
5	7	-4.0089171582464324580016512585964432909883144408807402330430(-24) ...
5	8	-1.6267872202198776712860024734866969196014335243663121425423(-27) ...
6	2	1.61508116616705485066326829316589367443121196123113678467541(-8) ...
6	3	1.53530343588928080283456861104632555466175700586234793665716(-13) ...
6	4	2.99324100743804909283239785893475963049642868205678987129596(-18) ...
6	5	7.41916124138504344746395187399890095182210117294534971585873(-23) ...
6	6	2.05602063767548079577453001329402669576701618478791822725187(-27) ...
6	7	6.06313176438106233958227084211400755691448415132455986317069(-32) ...
6	8	1.85800289034470997723167773802340218486625624481826862206504(-36) ...

4. APPLICATIONS

4.1. **Geometric Series.** By integrating [18, (1.511)]

$$(34) \quad \log(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n}$$

TABLE 7. Series of the form (39).

s	$\sum_{n=2}^{\infty} 1/[n^s(n-1)]$
1	1.
2	3.550659331517735635275848333539748107810500987932015622(-1) ...
3	1.530090299921792781278466718425248200160638064527026804(-1) ...
4	7.068579628104108661184297530135691724131285453397577278(-2) ...
5	3.375804113767116028047748884432274918423193503206296081(-2) ...
6	1.6414979153222020565955905340222128241444499920939897(-2) ...
7	8.065701771299193726162009203605461682550884433970692555(-3) ...
8	3.988345573354854347476770694952996423590093784120672226(-3) ...
9	1.979952747272639929624001462540935937984242389231915677(-3) ...
10	9.853776194545545924780425622219189319647108247543984198(-4) ...

we obtain the generating function [18, (1.513.5)],

$$(35) \quad x + (1-x) \log(1-x) = \sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}.$$

In the limit $x \rightarrow 1$ we find [18, (0.141)]

$$(36) \quad \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1.$$

Larger powers of the first factor of the denominator display partial sums of zeta functions through partial fraction decomposition,

$$(37) \quad \sum_{n=2}^{\infty} \frac{1}{n^2(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n(n-1)} - \frac{1}{n^2} \right) = 2 - \frac{\pi^2}{6};$$

$$(38) \quad \sum_{n=2}^{\infty} \frac{1}{n^3(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n(n-1)} - \frac{1}{n^2} - \frac{1}{n^3} \right) = 3 - \frac{\pi^2}{6} - \zeta(3);$$

$$(39) \quad \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = s - \sum_{l=2}^s \zeta(l); \quad s \geq 1.$$

Examples of these numbers are collected in Table 7, A152416 and A152419. Their sum is

$$(40) \quad \sum_{s=1}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^s} \frac{1}{n-1} = \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \zeta(2) = \frac{\pi^2}{6}.$$

Definition 4. *Restriction of the summation in (39) to k -almost primes defines a set of constants $B_{k,s}$ [19],*

$$(41) \quad B_{k,s} \equiv \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s(n-1)} = \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \sum_{l=0}^{\infty} \frac{1}{n^{s+1+l}} = \sum_{l=0}^{\infty} P_k(s+1+l).$$

TABLE 8. Some values of $B_{k,1}$. In accordance with (36), the series limit of the partial sums is 1 as $k \rightarrow \infty$.

k	$B_{k,1} = \sum_{n, \Omega(n)=k} 1/[n(n-1)]$
1	.77315666904979512786436745985594239561874133608318604831100606 ...
2	.17105189297999663662220256437237421399124661203550059749107997 ...
3	.41920339281764199227805032233471158322784525420828606710238790(-1) ...
4	.10414202346301156141109353888171559234184072973208943335673068(-1) ...
5	.25944317032356863609340108179412019910406474149863463566912649(-2) ...
6	.64712678336846601104554817217635310331423959328614964903156640(-3) ...
7	.16154547889045106884023528793253084539703632404976961733195128(-3) ...
8	.40350403394466614988860237144035458196679194641891917284153345(-4) ...
9	.10082343610557897391498490786448232831594447756388459395195738(-4) ...
10	.25198413274347214707213045392269003455525827323722877742607328(-5) ...
11	.62985737261498933173999701960384580565617680181013417876197437(-6) ...
12	.15745036385232517679881727385782020683287536986374137174089714(-6) ...
13	.39360719611599520681959076312200454332371020210585755644656168(-7) ...
14	.98399321710455906992308452734477378441132531283484928659273734(-8) ...
15	.24599505388932024146078978227171922132133360668716211893213680(-8) ...
16	.61498340075560680170866561648700452467948343944318880698907151(-9) ...
17	.15374530193448859466794967919224844842656125026606949920935630(-9) ...
18	.38436254839132442407411486355109175383764384816988280391113599(-10) ...
19	.96090546444389183823411185748926311892868554398657649851752505(-11) ...
20	.24022625018522170257126899356289067549236119426157474416944241(-11) ...
21	.60056547765687746047761256134140717295823654635402409654484759(-12) ...
22	.15014135061632779005230691663603510295794137329809014913244327(-12) ...
23	.37535335268553839790798447165432785944777132136305316859771955(-13) ...
24	.93838335149705572793424612556180553285473359587783011738232439(-14) ...
25	.23459583405297725332944711004418618278736884444821387508950778(-14) ...

This reduction to a geometric series and sum over the P_k has been used to calculate Table 8. This definition introduces an analog to (14),

$$(42) \quad \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = \sum_{k=1}^{\infty} B_{k,s}.$$

Remark 9. $B_{1,1}$ is A136141, calculated by Cohen [13]. $B_{1,2}$ is A152441. $B_{2,1}$ is A152447.

Projection of (39) onto the n of a fixed $\Omega(n)$ yields

$$(43) \quad B_{k,s} = B_{k,1} - \sum_{l=2}^s P_k(l).$$

The benefit of this formula is that the $B_{k,s}$ can be derived from $B_{k,1}$ without accumulating the individual terms of the geometric series proposed in (41), reaching back to Tables 1 and 4 instead. Consider for example $B_{3,2} \approx 0.003404 = B_{3,1} - P_3(2) \approx 0.041920 - 0.038516$ in Table 9.

The square-free variant of (41) is

TABLE 9. Some values of $B_{k,s}$. The series limits of the partial sums $\sum_k B_{k,s}$ are in Table 7.

k	s	$B_{k,s} = \sum_{\Omega(n)=k} 1/[n^s(n-1)]$
1	2	.32090924900872962935782409502369446144550999284329362657458713 ...
2	2	.30291458630973248399451638958496960216327336030620333566930993(-1) ...
3	2	.34041462990695583149671096054107628838843579424470347730472318(-2) ...
4	2	.40476614481790532069851037008630456765933446284259484392905784(-3) ...
1	3	.14614660970928609293471078035798776047009787091714433668591512 ...
2	3	.64854251582012887207556831056607595873951575502718767213033361(-2) ...
3	3	.35478421673524536821901075833145988403566339382745677940994463(-3) ...
4	3	.20861610202178413235709527570020237570255452209283788001114446(-4) ...
5	3	.12661340580586229820433777990228912341234438356263326260613819(-5) ...
1	4	.69153469945039247992091484424829890308056811202301146420977111(-1) ...
2	4	.14907506895639490854788090560814702648930997020410089927942399(-2) ...
3	4	.40356719902303626036886094140740676728492698470022769157503418(-4) ...
4	4	.11819765739964985563009679542861280192566527050014484191210481(-5) ...
5	4	.35812310330127538835013908172912057990127682625080625278844028(-7) ...
1	5	.33398452461114990859273241885974179176359534475649814730884436(-1) ...
2	5	.35473826470759431896325286534569759919935164601490043664281503(-3) ...
3	5	.47794665316209349180059179127476174264255324471931883104976848(-5) ...
4	5	.6987107560235151423551110353718404446313416171798954710770233(-7) ...
5	5	.10577117291999709417355375910626719164172948014247187268185467(-8) ...
1	6	.16328365610478477905139568619914779966773592601105569997721067(-1) ...
2	6	.86031097146184686541514125731610045619472901302328472349204898(-4) ...
3	6	.57819099766026370361808361682441463146581149531325986267386085(-6) ...
4	6	.42224059396278682956248009245427510821206201028665096074007060(-8) ...
5	6	.31946136165040413132735812808230110887705358254036598872046674(-10) ...

Definition 5.

$$(44) \quad B_{k,s}^{(\mu)} \equiv \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{\mu(n)}{n^s(n-1)}.$$

As in (41), there is a representation generated by expansion as a geometric series, and another one from the decomposition in partial fractions:

$$(45) \quad B_{k,s}^{(\mu)} = \sum_{l=0}^{\infty} P_k^{(\mu)}(s+1+l) = B_{k,1}^{(\mu)} - \sum_{l=2}^s P_k^{(\mu)}(l).$$

Explicit values follow in Tables 10 and 11. The special values of

$$(46) \quad B_{1,s}^{(\mu)} = -B_{1,s}$$

can be read off Table 8 and 9. Because the squared primes are those 2-almost primes which are not square-free, constants like

$$(47) \quad \sum_p \frac{1}{p^{2s}(p^2-1)} = \sum_{l=0}^{\infty} P(2(1+s+l)) = B_{2,s} - B_{2,s}^{(\mu)}$$

can be extracted subtracting values of Tables 9 and 11.

TABLE 10. Some values of (44) at $s = 1$.

k	$(-1)^k B_{k,1}^{(\mu)} = \sum_{\Omega(n)=\omega(n)=k} 1/[n(n-1)]$
1	.77315666904979512786436745985594239561874133608318604831100606 ...
2	.71606015364062950689014905233278570032977577496764766996881566(-1) ...
3	.37641725351677739987897144642934934884513171678284733095361749(-2) ...
4	.10533241426370309073189561057615806202677017536174797996070137(-4) ...
5	.16623463646913663848631359812999970758043030193325500480455497(-5) ...
6	.16150965195007452635819510602638099761518437046602308615745291(-7) ...
7	.10379657945831823210405127184359674418404044704893780297641014(-9) ...
8	.46608164350339032665792725000352856059530910663817211356715781(-12) ...
9	.15257916508734074179181916995271562612876849081982812066031366(-14) ...
10	.37674605405462816954865036936087328930035659282686679768832686(-17) ...
11	.72146307104358813058965067637397207142589348895613246863457289(-20) ...
12	.10966184934068789212128440266173951091947363648217302666195170(-22) ...
13	.13491272590180303187806861812072181049977941362569266836182658(-25) ...
14	.13659203913426921565993991785542457997100724146742659833750046(-28) ...
15	.11544550299305819020362074848337321347361648310620287413349137(-31) ...
16	.82468163663946083189906712482357029072394743693281076128206927(-35) ...
17	.50333187700965107357204271671076913932386822516030741929080100(-38) ...
18	.26498528956530623344847668001272588778001122254618728853331610(-41) ...
19	.12135579687796160737893591360857756602310726382140060310426006(-44) ...
20	.48713430940243727394794760257290243400665149725603953693181101(-48) ...
21	.17255587625660721457243893666076741199905259181309475423500537(-51) ...
22	.54270137257558710718858909894832920484581333704208745878297310(-55) ...
23	.15238881562580861279422356191754120203296455021672076901780409(-58) ...
24	.38397551449809407896292097416496474894690864603905322011328373(-62) ...
25	.87221420134797085669777164633365058358980257342379119911641791(-66) ...

4.2. Hurwitz Zeta Decompositions.

Definition 6. *The Hurwitz Zeta Function is*

$$(48) \quad \zeta(s, a) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \sum_{n=1}^{\infty} \frac{1}{(n+a-1)^s}; \quad \Re s > 1, \quad \Re a > 1.$$

On the trot, we project this sum onto the subspaces of k -almost primes, too,

Definition 7. *(Hurwitz Prime and Almost-Prime Zeta Functions)*

$$(49) \quad P_k(s, a) \equiv \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{(a-1+n)^s}; \quad P_k(s, 1) = P_k(s).$$

$$(50) \quad P_k^{(\mu)}(s, a) \equiv \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{\mu(n)}{(a-1+n)^s}; \quad P_k^{(\mu)}(s, 1) = P_k^{(\mu)}(s).$$

This partitions (48) into

$$(51) \quad \zeta(s, a) = \frac{1}{a^s} + \sum_{k=1}^{\infty} P_k(s, a),$$

TABLE 11. Some values of $(-1)^k B_{k,s}^{(\mu)}$.

k	s	$(-1)^k B_{k,s}^{(\mu)}$
2	2	.78387207792864074088832757525591864200993612067276933376705990(-2) ...
3	2	.67928371714938620394179831898825372859777454798046059788912729(-4) ...
4	2	.21490577139328324666260111005953570208961711778930701363329944(-6) ...
5	2	.26416111168510261601673859617237932837795292029557704345022153(-9) ...
2	3	.11027741571509606843209931657823850007531246009234812252063101(-2) ...
3	3	.17797204685886808724500022597142572187555600706419537190829226(-5) ...
4	3	.73257817973354076886116612880679583412683957083696886124052352(-9) ...
5	3	.76323430497840111065747811948812031341440309840660902575023325(-13) ...
2	4	.16950505503145160956755825888617620122821126476435449772990259(-3) ...
3	4	.52574659216450409428743432610083961488375889775555842142889903(-7) ...
4	4	.29750108591388430703115241596312267861630644973967306755420261(-11) ...
5	4	.26684984687313228427857617894414570413989159871977420443245736(-16) ...
2	5	.27096204612077060907852768298142951836335754023449262132044308(-4) ...
3	5	.16344646176240525767383244968461834157114119071517352759281108(-8) ...
4	5	.13043647405152222773308067259925713545003355242564896189378205(-13) ...
5	5	.10128782422925704028633932747239792053264472723250762731720244(-19) ...
2	6	.44155070852131080127626208491363262363892772582004900125801062(-5) ...
3	6	.52141702169123237830700571831415136754126549873293803568336271(-10) ...
4	6	.59376215895380325925646622617419720223297866728632203157065385(-16) ...
5	6	.40105446162489087900384620760650048636932117449331297950429083(-23) ...

generalizing (14).

Remark 10. By series expansion [18, (1.112.1)], the sum rule associated with (41) is

$$(52) \quad B_{k,s} = \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{(n-1)^{1+s}} \frac{1}{1 + \frac{1}{n-1}} = \sum_{l=0}^{\infty} (-1)^l P_k(1+s+l, 0).$$

The reduction of (49) to the Prime Zeta Functions is obtained by the binomial expansion [18, (1.110)][7, Entry 22]

$$(53) \quad P_k(s, a) = \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} \frac{1}{\left(1 - \frac{1-a}{n}\right)^s} = \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} \sum_{l=0}^{\infty} \binom{-s}{l} (-1)^l \frac{(1-a)^l}{n^l} \\ = \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} \sum_{l=0}^{\infty} \frac{(s)_l}{l!} \frac{(1-a)^l}{n^l} = \sum_{l=0}^{\infty} \frac{(s)_l}{l!} (1-a)^l P_k(s+l).$$

$$(54) \quad P_k^{(\mu)}(s, a) = \sum_{l=0}^{\infty} \frac{(s)_l}{l!} (1-a)^l P_k^{(\mu)}(s+l).$$

Pochhammer's symbol $(s)_l \equiv \Gamma(s+l)/\Gamma(s)$ is introduced to simplify the notation [1, (6.1.22)]. Brute-force accumulation of partial sums over l for the case $a = 0$

TABLE 12. Some values of $P_k(s, 0)$. Cohen has reported $P_1(2, 0)$ [13], which is A086242 [32].

k	s	$P_k(s, 0) = \sum_{\Omega(n)=k} 1/(n-1)^s$
1	2	1.37506499474863528791725313052243969917959996017531745870918933 ...
2	2	.209788323940019492755368602469189236268613932921851343752817089 ...
3	2	.457250649473356179509462896789867962239663545084304304104898823(-1) ...
4	2	.108410864467466394130511466215222764287414954775183405819497437(-1) ...
5	2	.264509027543661792197346769155986673457680151009439096206096105(-2) ...
1	3	1.14752909775858004693328380628213040164476473552511225527582412 ...
2	3	.497610511326665981875950866377952405841340774755637814783395177(-1) ...
3	3	.425728972912996225107947200519463804614525467386937187206041965(-2) ...
4	3	.450337561601661496970504401383245773976032018827147548523156327(-3) ...
5	3	.519996368729267202444682115275828429696087549531802544406653646(-4) ...
1	4	1.06736011227157169811527402065258703893525859304550836811508241 ...
2	4	.144251867050125347206321771385325774655732916871746611337540069(-1) ...
3	4	.511594573334901127268316650383244703997517641565236190813247750(-3) ...
4	4	.248732308184070333068620411510121468878257641611466236659372052(-4) ...
5	4	.138025203867843406309850933144822342517665004454740887033517806(-5) ...
1	5	1.03237100597834196585177592063868294503482496931776985434695475 ...
2	5	.448814553317860895890452521422141457224197062051853943952316742(-2) ...
3	5	.670525710706644570099917906002173405129522585443401784675937527(-4) ...
4	5	.150947118336184838341631597141043715807449152523875014126944996(-5) ...
5	5	.403832718324949534495289315322822843326177902298377065444183828(-7) ...
1	6	1.01589201139972411006675918457325510103473578053731168636932226 ...
2	6	.144181860839192758202651548654521805154418089925454726071029138(-2) ...
3	6	.913523107817850408748985104163510342741650140340861620051453925(-5) ...
4	6	.954942045591842308217096235482438389525995484180595401298115858(-7) ...
5	6	.123322395562635072773966382461892764518565670709235951708582994(-8) ...

yields Table 12. Resummation in (49) imposes the sum rule

$$(55) \quad \sum_{k=1}^{\infty} P_k(s, 0) = \zeta(s).$$

The derivative of (53) with respect to s is

$$(56) \quad \begin{aligned} \frac{\partial}{\partial s} P_k(s, a) &= P'_k(s, a) = - \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{\log(a-1+n)}{(a-1+n)^s} \\ &= \sum_{l=0}^{\infty} \frac{(s)_l (1-a)^l}{l!} [(\psi(s+l) - \psi(s)) P_k(s+l) + P'_k(s+l)] \end{aligned}$$

in terms of digamma functions ψ [1, (6.3)]. Cases with $a = 0$ are illustrated by Table 13.

4.3. Logarithmic Functions. Repeated integration of (35) with respect to x yields a ladder of functions with increasing order $l + 1$ of the polynomial of n

TABLE 13. Some absolute values of (56) at $a = 0$.

k	s	$ P'_k(s, 0) = \sum_{\Omega(n)=k} \log(n-1)/(n-1)^s$
1	2	.412038626948453592989536727886919593108693955993272284789334253 ...
2	2	.349406402843729094021858840552328411755268945853567444645947573 ...
3	2	.122053877557071229252860135234470821144298661274431502394539860 ...
4	2	.381324604512840984602505783616974026559738868834901273236913385(-1) ...
5	2	.113453301459081906536851293804267570546096249239456318213977158(-1) ...
1	3	.122491994469611894418110664126546306148971700364574609076227728 ...
2	3	.647147749533078072180333326031334769093826259630394154513521204(-1) ...
3	3	.935118212996044867853444829503103960682129910900028893869639531(-2) ...
4	3	.134268150501925211286561753670177005489358798054445521080127694(-2) ...
5	3	.193322938011967430559782708011999395264147756118102143367684017(-3) ...
1	4	.505703011282046113580449366113852907995938099426370524892239106(-1) ...
2	4	.172097052934328020906454081592687823346699820793905416728793462(-1) ...
3	4	.105543731770635366656467702388193888381593405142250008490173220(-2) ...
4	4	.705255178358882713248498070972826082493637781890072290188334302(-4) ...
5	4	.492194845815858173046505381936518534428557359827933872744842926(-5) ...
1	5	.232833728972359609747954746653486493466064393424300179653066541(-1) ...
2	5	.515141863816847304246565784890839865650309296489970884531366999(-2) ...
3	5	.134651494862962805588752454576895989572879746878237372363930174(-3) ...
4	5	.419081737714005222851533545272768768198324374375815699071937904(-5) ...
5	5	.141550001290993202090903719190199844283347760149056451512440612(-6) ...
1	6	.112107071669452487391848834784015585352560002552358622406258928(-1) ...
2	6	.162309029366737494046263910398940347901702033172772596446220636(-2) ...
3	6	.181009248757925488622702113921921885223627925005301730877648970(-4) ...
4	6	.262385322411092391533263049109867509130968921137179066906794859(-6) ...
5	6	.428617889403977448189220166642396011211315801456016154892551835(-8) ...

in the denominator [18, (1.513.7)][24],

$$(57) \quad x + (1-x)^2 \ln(1-x) = \frac{3x^2}{2} - 2 \sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+1)(n+2)};$$

$$(58) \quad x + (1-x)^3 \ln(1-x) = \frac{5x^2}{2} - \frac{11x^3}{6} + 6 \sum_{n=1}^{\infty} \frac{x^{n+3}}{n(n+1)(n+2)(n+3)};$$

$$(59) \quad x + (1-x)^4 \ln(1-x) = \frac{7x^2}{2} - \frac{13x^3}{3} + \frac{25x^4}{12} - 24 \sum_{n=1}^{\infty} \frac{x^{n+4}}{n(n+1)(n+2)(n+3)(n+4)};$$

$$(60) \quad x + (1-x)^l \ln(1-x) = \sum_{i=2}^l \tau_{i,l} x^i - (-1)^l l! \sum_{n=1}^{\infty} \frac{x^{n+l}}{n(n+1) \cdots (n+l)}.$$

The rational coefficients of the auxiliary polynomials obey the recurrence

$$(61) \quad \tau_{2,l} = l - \frac{1}{2}, \quad l \geq 2; \quad i\tau_{i,l} = (-1)^i \binom{l-1}{i-1} - l\tau_{i-1,l-1}, \quad i \geq 3.$$

$i! \tau_{i,l}$ are tabulated in A067176 and A093905.

TABLE 14. Logarithmic sums of k -almost primes defined in (62).

k	l	$L_{k,l}$
1	1	0.26434004176002164673174824590306461492528050270047150992644 ...
1	2	0.61210909600861648810973500903074917744536817685379132220140 ...
1	3	0.83162031622701364585608159976581294069621021299936900413624 ...
1	4	0.97814589475979949499502571065792051537852172306539544173835 ...
2	1	0.07483176753552750888163142884027057172300662738030777281173 ...
2	2	0.20274600679905285623473576314229971108548607276186973818502 ...
2	3	0.30956716856446670438488608714185233323846931207915051554876 ...
2	4	0.39967077199200952417631057083019796152634867816759135315832 ...
3	1	0.01979445800870394214182491696866714839521363828266446266352 ...
3	2	0.05673037199050316150740109374211659920215472144741775457277 ...
3	3	0.09078044492621127414186177360393579101033447804508695808347 ...
3	4	0.12224056980762694782137972349781919346596885467095976821370 ...

Substitution of inverse k -almost primes for x in (60) followed by summation defines constants $L_{k,l}$ which generalize (4),

$$\begin{aligned}
 L_{k,l} &\equiv \sum_{\Omega(n)=k} \left[\frac{1}{n} + \left(1 - \frac{1}{n}\right)^l \ln \left(1 - \frac{1}{n}\right) \right] \\
 (62) \quad &= \sum_{s=2}^l \tau_{s,l} P_k(s) - (-1)^l l! \sum_{s=1}^{\infty} \frac{1}{s(s+1) \cdots (s+l)} P_k(s+l),
 \end{aligned}$$

placed in Table 14. At $l = 1$, partial fraction decomposition of $1/[s(s+1)]$ in (62) implies that $L_{1,1}$ plus the value at $u = 1$ in Table 2 equals the value in (5). Series of the form $\sum_{s \geq 1} P_k(s+l)/\prod_{i=0}^l (s+i)$ are quickly accessible by subtracting the τ -dependent terms of (62) with the aid of Table 4. Varieties with non-contiguous factors in the denominator can be reduced to the $L_{k,l}$ basis by partial fraction synthesis: examples of this approach with one missing factor $s+2$ or two missing factors s and $s+2$ on the left hand side are

$$(63) \quad \frac{1}{s(s+1)(s+3)} = \frac{1}{s(s+1)(s+2)} - \frac{1}{s(s+1)(s+2)(s+3)}$$

and

$$(64) \quad \frac{1}{(s+1)(s+3)} = \frac{1}{s(s+1)} - \frac{3}{s(s+1)(s+2)} + \frac{3}{s(s+1)(s+2)(s+3)}.$$

Remark 11. The case of $x = 1/2$ in (34) is decomposable in k -almost primes as well [17, 9]

$$(65) \quad \log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} + \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ \Omega(n)=k}}^{\infty} \frac{1}{n2^n}.$$

The dominating components are

$$\begin{aligned}
 (66) \quad & \sum_{\substack{n=1 \\ \Omega(n)=k}}^{\infty} \frac{1}{n2^n} \\
 &= 0.174087071760979362471993316621554442658749500081033068401 \dots (k=1) \\
 &= 0.185502662799497065892654852882047774301689318692751270328(-1) \dots (k=2) \\
 &= 0.508886353469978309538645063262227959518399790876499567910(-3) \dots (k=3) \\
 &= 0.956158270230416532955864299570858933417801804520522813509(-6) \dots (k=4)
 \end{aligned}$$

These sums generated from $|x| < 1$ converge rapidly on their own and do not need the acceleration strategies outlined in the first sections.

Remark 12. The limits $x \rightarrow 1$ in (57)–(60) generalize (36) [34, (p. 42)]

$$(67) \quad \frac{1}{k \cdot k!} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2) \cdots (n+k)}.$$

Restricted summation over k -almost primes in the spirit of (41) would split these into another basic type of constants.

5. SUMMARY

Almost-prime zeta functions have been defined by restriction of the summation of the standard definition of zeta functions to k -almost primes. Their values can be bootstrapped from a multinomial overlay of the values of the ordinary prime zeta functions. Efficient schemes to compute the latter have been employed to calculate series summed over k -almost primes of some basic inverse polynomials to high accuracy.

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34. Daniel Zwillinger (ed.), *CRC standard mathematical tables and formulae*, 31 ed., Chapman & Hall/CRC, Boca Raton, FL, 2003, E: the lower limit in eqs. (11)–(14) on page 42 ought be $k = 1$, not $k = 0$.

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